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Asymptotically plane wave spacetimes and their actions

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Abstract: We propose a definition of asymptotically plane wave spacetimes in vacuum gravity in terms of the asymptotic falloff of the metric, and discuss the relation to previously constructed exact solutions. We construct a well-behaved action principle for such spacetimes, using the formalism developed by Mann and Marolf. We show that this action is finite on-shell and that the variational principle is well-defined for solutions of vacuum gravity satisfying our asymptotically plane wave falloff conditions.

Keywords: Penrose limit and pp-wave background, Black Holes, Classical Theories of Gravity.

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1. Introduction

Plane waves are interesting from a variety of different points of view (see [1] for a review and further references): they provide a rich class of exact solutions to Einstein's equations, which describe the neighbourhood of a null geodesic in any geometry by the Penrose limit. They also include some maximally supersymmetric solutions of supergravity, on which the string worldsheet theory is exactly solvable. Since the seminal work of [2] on the Penrose limit of $\mathrm{AdS}_5 \times S^5$, the string theory on the maximally supersymmetric plane wave has also been of intense interest as an example of holography [3]. The spectrum of strings on the plane wave is related to the spectrum of a quantum mechanical system obtained from the dual CFT on the boundary of the AdS_5 space. This connection provides stringy tests of the AdS/CFT correspondence, and has significantly deepened our understanding of this duality.

However, our understanding of holography for the plane wave is still incomplete: the duality is more indirect than AdS/CFT, since the dual quantum mechanics is obtained from the theory on the boundary of AdS, whereas the Penrose limit which gives rise to the plane wave focuses on a region at the center of AdS. Although a well-defined notion of the boundary of the maximally supersymmetric plane was obtained by conformal compactification in [4], and this boundary turns out to be one-dimensional, a direct connection between the string theory on this plane wave and a theory living in some sense on its asymptotic boundary has not yet been constructed. As a result, it has not been possible to extend the results of [3] to discuss a holographic duality for general plane waves.

One approach to deepening our understanding of the duality for plane waves is to construct asymptotically plane wave spacetimes, and to look for interpretations of these spacetimes in field theory terms. In particular, it is clearly interesting to construct asymptotically plane wave black holes and black strings. The construction of such solutions has been discussed in [5 − 10]. The asymptotic structure of plane waves has also been discussed from a general point of view, using the causal completion of the spacetime, in [11 − 13].

Another interesting recent development for holography was the construction of a well-behaved action principle for asymptotically flat spacetimes in [14] (see also [15, 16]), which was argued in [17] to provide an approach to defining a holographic dual to asymptotically flat space. This was extended to study holography for linear dilaton spacetimes in [18].

Our aim in this paper is to similarly construct an action principle for asymptotically plane wave spacetimes, in the hope that this will shed some light on the issue of holography for plane waves. Our results may also be useful for other investigations of asymptotically plane wave spacetimes: for example, these methods can be used to calculate conserved quantities.

To discuss the action for asymptotically plane wave spacetimes, we first need a suitable notion of what it means for a spacetime to be asymptotically plane wave. In section 2, we propose a definition in terms of a set of falloff conditions on the metric at large spatial distances in directions orthogonal to the wave. We start by assuming that the components of the metric with indices along the spatial directions orthogonal to the wave fall off as $\mathcal{O}(r^{2-d})$, where r is a radial coordinate and d is the number of spatial directions orthogonal to the wave, corresponding to the influence of a localised source being spread over a $(d-1)$-sphere at large distances. We then need to determine the behaviour of the components of the metric with indices parallel to the wave; we use the linearised equations of motion to relate the falloff conditions of different components, by assuming that all components make contributions of the same order to each term in the Einstein equations. This fixes the falloff of the other components of the metric. We will show that the known solutions which asymptotically approach a vacuum plane wave [5 − 7] satisfy our falloff conditions.

We only study solutions of the vacuum Einstein equations; it would clearly be interesting to extend this to include matter, and in particular to supergravity. We will see that the black string solution of [9], which asymptotically approaches a plane wave solution in supergravity, does not satisfy our falloff conditions. The extension to include matter may therefore be non-trivial, as in the AdS case, where the presence of a scalar field can lead to the existence of more general AdS-invariant boundary conditions for the metric [19].

In section 3, we show that the definition of the action for vacuum gravity introduced in [14] can be applied to asymptotically plane wave spacetimes with our falloff conditions without significant modification. We demonstrate that the on-shell action is finite, and that the variational principle is well-defined. This provides confirmation that this is a useful definition of asymptotically plane wave, and provides another example where the counterterm approach of [14] is useful, suggesting that this approach to defining the gravitational action should have a broad applicability.

We will close the paper in section 4 with some comments and remarks. An open problem for the future is to apply this definition of the action to calculate the conserved quantities for the asymptotically plane wave spacetimes.

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2. Asymptotically plane wave falloff conditions

We consider asymptotically plane wave solutions in vacuum gravity. The plane wave solutions in $d+2$-dimensional vacuum gravity can be written in Brinkmann coordinates as

$$ds^{2(0)} = -2dx^+dx^- - \mu_{ij}(x^+)x^ix^j \left(dx^+\right)^2 + \delta_{ij}dx^idx^j, \tag{2.1}$$

where $i,j = 1,\ldots,d$, and $\mu_{ij}(x^+)$ are arbitrary functions subject only to $\delta^{ij}\mu_{ij}(x^+) = 0$, which ensures that the solution satisfies the vacuum equations of motion. The coordinates in the plane wave solution split into two coordinates $x^\pm$ along the direction of the wave, and the spatial coordinates x^i in the directions orthogonal to the wave. In the spatial directions, we will use both Cartesian coordinates x^i, and polar coordinates r, θ^a, $a = 1,\ldots(d-1)$:

$$\delta_{ij}dx^idx^j = dr^2 + r^2\hat{h}_{ab}d\theta^a d\theta^b, \tag{2.2}$$

where $\hat{h}_{ab}$ is the metric and θ^a are the coordinates on the unit $(d-1)$-sphere S^{d-1}.

A general asymptotically plane wave spacetime will have a metric $g = g^{(0)} + g^{(1)}$, where $g^{(1)}$ will have some suitable falloff conditions at large distance. We will focus on studying the falloff conditions at large radial distance in the directions orthogonal to the wave. In the spatial direction that the wave is travelling in, we will consider either perturbations which are independent of x^-, like the wave itself, or perturbations which fall off at large x^-, but we will not explicitly specify the falloff conditions in this direction.[1]

Considering first metrics which are independent of x^-, we specify the falloff conditions at large r by making two assumptions. First, we assume that the spatial components (in the above Cartesian coordinate system) $g^{(1)}_{ij} \sim \mathcal{O}\left(r^{2-d}\right)$. These are the same falloff conditions as for the spatial components of an asymptotically flat metric in $d+1$ dimensions. This seems appropriate because we would expect a perturbation which is independent of x^- to correspond to the effect of a source which is extended along the direction of the wave, but localised in the transverse spatial directions, so its effect at large r should be diluted by spreading on the S^{d-1}.

To fix the falloffs of $g_{\pm\pm}$, $g_{\pm i}$, we make a second assumption, that all components make contributions of the same order to each term in the Einstein equations.[2] This is essentially a genericity assumption, so it should be appropriate for finding the general falloff conditions on metric components. In vacuum gravity, the linearised equations of motion are $R^{(1)}_{\mu\nu} = 0$, where [20]

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}g^{(0)\rho\sigma}\,\nabla^{(0)}_\rho\,\nabla^{(0)}_\sigma g^{(1)}_{\mu\nu} - \frac{1}{2}g^{(0)\rho\sigma}\,\nabla^{(0)}_\mu\,\nabla^{(0)}_\nu g^{(1)}_{\rho\sigma} + g^{(0)\rho\sigma}\,\nabla^{(0)}_\rho\,\nabla^{(0)}_{(\mu}\, g^{(1)}_{\nu)\sigma}. \tag{2.3}$$

[1] This is similar to the treatment of linear dilaton spacetimes in [18], where the falloffs in the directions along the brane were not explicitly treated.

[2] We will not attempt to fully exploit the information in the asymptotic Einstein equations; we just use them to determine a set of falloff conditions. The consistency of our falloff conditions with the dynamical equations of motion is demonstrated by verifying that the solutions we consider in the next subsection satisfy our falloff conditions.

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The idea of our assumption is that the cancellations which give $R^{(1)}_{\mu\nu} = 0$ should generically involve all the terms in $R^{(1)}_{\mu\nu}$. The contribution of $g^{(1)}_{ij}$ to (2.3) gives

$$R^{(1)}_{ij} \sim \mathcal{O}\left(r^{-d}\right), \quad R^{(1)}_{+i} \sim \mathcal{O}\left(r^{1-d}\right), \quad R^{(1)}_{++} \sim \mathcal{O}\left(r^{2-d}\right). \tag{2.4}$$

Because of the assumption that $g^{(1)}_{ij}$ is constant in x^-, it does not make any contribution to $R^{(1)}_{-i}, R^{(1)}_{+-}$ and $R^{(1)}_{--}$. Assuming the other terms in $g^{(1)}_{\mu\nu}$ produce effects at the same order determines

$$g^{(1)}_{++} \sim \mathcal{O}\left(r^{4-d}\right), \quad g^{(1)}_{+-} \sim \mathcal{O}\left(r^{2-d}\right), \quad g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d}\right), \tag{2.5}$$

$$g^{(1)}_{+i} \sim \mathcal{O}\left(r^{3-d}\right), \quad g^{(1)}_{-i} \sim \mathcal{O}\left(r^{1-d}\right). \tag{2.6}$$

With these falloffs, all terms also give

$$R^{(1)}_{-i} \sim \mathcal{O}\left(r^{-d-1}\right), \quad R^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d}\right), \quad R^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-2}\right). \tag{2.7}$$

The faster falloff conditions required for metric components with an x^- index arise because $g^{(0)--} \sim r^2$, so terms in a given component of $R^{(1)}_{ij}$ coming from $g^{(1)}_{--}$ have an extra factor of r^2 compared to terms coming from $g^{(1)}_{ij}$. Similarly, the less restrictive conditions on components with an x^+ index are due to the vanishing of $g^{(0)++}$.

If we consider the more general case, allowing the perturbation to depend on x^-, there will be additional terms in $R^{(1)}_{\mu\nu}$ involving derivatives ∂_-. These terms will also come with extra powers of r coming from $g^{(0)--}$. As a result, if we think of a general perturbation as composed of a part which is independent of x^- and a part which depends on x^-, the part which depends on x^- will be required to fall off more quickly than the constant part.[3] We find

$$\partial_- g^{(1)}_{ij} \sim \mathcal{O}\left(r^{-d}\right), \qquad \partial_- g^{(1)}_{+j} \sim \mathcal{O}\left(r^{1-d}\right), \qquad \partial_- g^{(1)}_{-j} \sim \mathcal{O}\left(r^{-d-1}\right), \tag{2.8}$$

$$\partial_- g^{(1)}_{++} \sim \mathcal{O}\left(r^{2-d}\right), \qquad \partial_- g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d}\right), \qquad \partial_- g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-2}\right), \tag{2.9}$$

and

$$\partial_-\partial_- g^{(1)}_{ij} \sim \mathcal{O}\left(r^{-d-2}\right), \qquad \partial_-\partial_- g^{(1)}_{+j} \sim \mathcal{O}\left(r^{-d-1}\right), \qquad \partial_-\partial_- g^{(1)}_{-j} \sim \mathcal{O}\left(r^{-d-3}\right), \tag{2.10}$$

$$\partial_-\partial_- g^{(1)}_{++} \sim \mathcal{O}\left(r^{-d}\right), \qquad \partial_-\partial_- g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d-2}\right), \qquad \partial_-\partial_- g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-4}\right). \tag{2.11}$$

[3] Even without this additional factor, the x^- dependent parts would be required to fall off faster than the constant parts. The situation is analogous to the solution for a localized source described in a cylindrical coordinate system, which involves

$$\frac{1}{(r^2+z^2)^{\frac{d-2}{2}}} \approx \frac{1}{r^{d-2}} - \frac{(d-2)z^2}{2r^d} + \cdots ,$$

so the z-dependent term falls off faster than the constant term at large r. The effect of $g^{(0)--}$ is to make these contributions fall off even more quickly in the plane wave background.

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We take the above constraints on the asymptotic falloff of the metric to define a class of asymptotically plane wave spacetimes.

Not all of these components of the metric carry independent physical information; by an appropriate diffeomorphism, we can set some of the components $g^{(1)}_{\mu\nu}$ to zero at large distance. In [18], this diffeomorphism freedom was fixed by choosing a Gaussian normal gauge, in which the components of $g^{(1)}_{\mu\nu}$ with radial indices are set to zero. In the present case, because the directions $x^\pm$ are singled out as special, it seems more convenient to us to choose a gauge in which

$$g^{(1)}_{+-} = g^{(1)}_{--} = g^{(1)}_{-i} = 0. \tag{2.12}$$

Because of the faster falloff conditions on the x^- components, the diffeomorphism which sets these components to zero will not modify the asymptotic falloff of the other components.

2.1 Comparison to known solutions

There have been a few papers on exact solutions of the Einstein equations which asymptotically approach a plane wave. These provide a useful check of our analysis: if we have an appropriate set of falloff conditions, they should be satisfied by these solutions. The first such solution was constructed in [5, 6], where a Garfinkle-Vachaspati transform was applied to a black string solution with a nontrivial scalar field to obtain an asymptotically plane wave black string,

$$\begin{aligned} ds^2_{str} &= -\frac{2}{h(r)}dx^+dx^- + \frac{f(r)+r^2(3\cos^2\theta-1)}{h(r)}(dx^+)^2 + (k(r)l(r))^2(dr^2+r^2d\Omega_2^2), \\ e^{4\phi} &= \frac{k(r)l(r)}{h^2(r)}, \end{aligned} \tag{2.13}$$

where

$$f(r) = 1+\frac{Q_1}{r}, \quad h(r) = 1+\frac{Q_2}{r}, \quad k(r) = 1+\frac{P_1}{r}, \quad l(r) = 1+\frac{P_2}{r}. \tag{2.14}$$

The presence of the scalar ϕ means that this is not a vacuum solution, but it becomes a vacuum solution at large r, and it is easy to check that our boundary conditions are satisfied. The solution is independent of x^-, and it has $g^{(1)}_{+-}$ and $g^{(1)}_{ij}$ going like $\mathcal{O}\left(r^{-1}\right)$, $g^{(1)}_{++}$ going like $\mathcal{O}\left(r\right)$, with the other components of $g^{(1)}_{\mu\nu}$ vanishing. We have written the string frame solution above but this statement will be true in either string or Einstein frame.

This was extended in [7] to construct a pure vacuum solution which is asymptotically plane wave, although it is not smooth in the interior:

$$ds^2 = \frac{1}{H(r)}\left[-2dx^+dx^- + f(r)(dx^+)^2 + \frac{H(r)^4}{r^4H'(r)^2}(dr^2+r^2d\Omega_2^2)\right], \tag{2.15}$$

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where

$$f(r) = 1 + \ln H(r) + \xi_2(x^+)\psi_2(r)(3\cos^2\theta - 1), \tag{2.16}$$

$$\psi_2(r) = (3r^2 + 2 + 3r^{-2})\left[\alpha_1 + \alpha_2 \ln\left(\frac{r-1}{r+1}\right)\right] + 6\alpha_2(r + r^{-1}), \tag{2.17}$$

$$H(r) = \left(\frac{r-1}{r+1}\right)^{\frac{2}{\sqrt{3}}}, \tag{2.18}$$

and α_1, α_2 are arbitrary constants and $\xi_2(x^+)$ is an arbitrary function of x^+. Again, it is easy to see that this satisfies our definition of asymptotically plane wave. The solution is independent of x^-, and it has $g^{(1)}_{+-}$ and $g^{(1)}_{ij}$ going like $\mathcal{O}\left(r^{-1}\right)$, $g^{(1)}_{++}$ going like $\mathcal{O}\left(r\right)$, with the other components of $g^{(1)}_{\mu\nu}$ vanishing.

In [8], a solution was obtained by T-duality from a black hole in a Gödel universe. This solution reduces to a plane wave when the black hole mass parameter is set to zero, but it is not asymptotically plane wave, as it has components $g^{(1)}_{ij}$ going like $\mathcal{O}\left(r^0\right)$ at large r, so the sphere is deformed asymptotically. Thus, it does not satisfy our definition, but this is unproblematic: we would not regard such a solution as a candidate for the appellation asymptotically plane wave.

Finally, another solution was obtained in [9] by a sequence of boosts and dualities known as the null Melvin twist. This is a solution in the common Neveu-Schwarz sector of the ten-dimensional superstring theories, and has

$$\begin{aligned} ds^2_{str} = &-\frac{f(r)(1+\beta^2r^2)}{k(r)}dt^2 - \frac{2\beta^2r^2f(r)}{k(r)}dtdy + \left(1 - \frac{\beta^2r^2}{k(r)}\right)dy^2 \\ &+\frac{dr^2}{f(r)} + r^2d\Omega_7^2 - \frac{\beta^2r^4(1-f(r))}{4k(r)}\sigma^2, \end{aligned} \tag{2.19}$$

$$e^\phi = \frac{1}{\sqrt{k(r)}}, \tag{2.20}$$

and

$$B = \frac{\beta r^2}{2k(r)}(f(r)dt + dy)\wedge\sigma, \tag{2.21}$$

where

$$f(r) = 1 - \frac{M}{r^6}, \quad k(r) = 1 + \frac{\beta^2 M}{r^4}, \tag{2.22}$$

and the one-form σ is given in terms of Cartesian coordinates x^i by

$$\frac{r^2\sigma}{2} = x^1dx^2 - x^2dx^1 + x^3dx^4 - x^4dx^3 + x^5dx^6 - x^6dx^5 + x^7dx^8 - x^8dx^7. \tag{2.23}$$

This solution is not vacuum, even at large distances, but at large r it approaches a plane wave which [9] call $\mathcal{P}_{10}$, which is the two-form equivalent of an electromagnetic plane wave. We can then write the metric as $g = g^{(0)} + g^{(1)}$, where $g^{(0)}$ is the metric of the pure plane wave $\mathcal{P}_{10}$, which can be obtained by setting $M = 0$ in the above solution.

This solution lies outside of the scope of our analysis, since it is not a solution of the vacuum Einstein equations, even asymptotically. However, we can still observe that

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this solution does not satisfy our asymptotic falloff conditions, as $g^{(1)}_{ij} \sim \mathcal{O}\left(r^{-4}\right)$, so our input assumption that $g^{(1)}_{ij} \sim \mathcal{O}\left(r^{2-d}\right)$ is not satisfied. That is, the spatial falloff of the metric is not behaving as we would expect based on a localised source, which presumably means that there are source terms coming from the two-form field B which extend into the asymptotic region, additional to those associated with the plane wave $\mathcal{P}_{10}$. In addition, the relation between the different coefficients is not the same as we had: if we define $x^+ = t+y$, $x^- = t-y$, we will have $g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-4}\right)$, but $g^{(1)}_{--} \sim \mathcal{O}\left(r^{-4}\right)$, and not $\mathcal{O}\left(r^{-6}\right)$ as we might have expected from the behaviour of $g^{(1)}_{ij}$. It is not clear whether we should regard this solution as asymptotically plane wave or not; it asymptotically approaches the plane wave metric $\mathcal{P}_{10}$, but more slowly than we would expect. In particular, the slow falloff of the spatial components $g^{(1)}_{ij}$ is likely to make it difficult to define a finite action principle for such solutions. It would be very interesting to extend our analysis below to include form fields so that this case could be directly addressed.

2.2 Conformal structure

We have given a definition of asymptotically plane wave spacetimes above, focusing on the behaviour of the solution at large r. Our decision to focus on the behaviour at large r is inspired in part by the previously-known exact solutions, which approach a plane wave only at large r, and by our interest in the construction of an appropriate action principle, where it is the $r =$ constant boundary which is expected to be problematic.

In special cases, however, we could take a different approach, and define asymptotically plane wave spacetimes in terms of the existence of a suitable conformal completion. This would be closer in spirit to the usual treatments of asymptotic flatness. We will not develop this approach here; we simply want to make some remarks pointing out that it is really quite different to the approach we are taking.

In [4], a conformal completion was constructed for the maximally supersymmetric plane wave, for which the metric is

$$ds^2 = -2dx^+dx^- - r^2(dx^+)^2 + dr^2 + r^2 d\Omega_7^2, \tag{2.24}$$

where $d\Omega_7^2$ denotes the unit metric on S^7. The conformal completion is obtained by making a coordinate transformation to rewrite this metric as a conformal factor times the metric on the Einstein static universe,

$$ds^2 = \frac{1}{|e^{i\psi} - \cos\alpha e^{i\beta}|^2}(-d\psi^2 + d\alpha^2 + \cos^2\alpha d\beta^2 + \sin^2\alpha d\Omega_7^2). \tag{2.25}$$

We thus see that the conformal boundary of this plane wave lies at $\alpha = 0$, $\psi = \beta$, and is a one-dimensional null line in the Einstein static universe. The explicit coordinate transformation is

$$r = \frac{\sin\alpha}{2|e^{i\psi} - \cos\alpha e^{i\beta}|}, \tag{2.26}$$

$$\tan x^+ = \frac{\sin\psi - \cos\alpha\sin\beta}{\cos\psi - \cos\alpha\cos\beta}, \tag{2.27}$$

$$x^- = \frac{1}{2}\left(\frac{\sin\psi + \cos\alpha\sin\beta}{\cos\psi - \cos\alpha\cos\beta} - r^2\tan x^+\right). \tag{2.28}$$

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The point we want to stress is that when we approach the conformal boundary $\alpha = 0$, $\psi - \beta = 0$ along a generic direction, say $\alpha = \gamma(\psi - \beta)$ for some constant γ, r remains finite. In these generic directions, it is x^- which diverges. Thus, controlling the behaviour as $r \to \infty$ in a spacetime which asymptotically approaches this plane wave will give little information about whether there exists a conformal completion with (in some suitable sense) "the same structure" as for the pure plane wave. Rather, it is the behaviour at large x^- that one would have to study in detail to see if a suitable conformal completion exists.

Thus, the definition of asymptotically plane wave we have introduced is different in character from a definition based on conformal structure. If a definition based on conformal structure could be developed, it would presumably be suitable for addressing different questions from those which can be addressed with our definition. We would also remark that the above analysis suggests that the known exact solutions, which have a deformation away from the plane wave which is independent of x^-, are unlikely to qualify as asymptotically plane wave with respect to such a conformal definition of asymptotically plane wave.

3. Action for asymptotically plane wave spacetimes

We have put forward a definition of asymptotically plane wave spacetimes, using the linearised equations of motion to relate the falloff of different components. In this section, we give our main result, constructing an appropriate action principle for this class of spacetimes. We construct our action principle following Mann and Marolf [14], who recently introduced a new approach to specifying a well-defined action principle for vacuum gravity for asymptotically flat spacetimes.

For the asymptotically flat case, the action is [14]

$$S = -\frac{1}{16\pi G}\int_M \sqrt{-g} R d^D x - \frac{1}{8\pi G}\int_{\partial M} \sqrt{h} K d^{D-1}x + \frac{1}{8\pi G}\int_{\partial M} \sqrt{h}\hat{K} d^{D-1}x, \qquad (3.1)$$

where g is the determinant of the bulk metric, h is the determinant of the bulk metric pulled back to the boundary, R is the Ricci scalar, and $K = h^{\alpha\beta}K_{\alpha\beta}$ is the trace of the extrinsic curvature on the boundary. The final term is a new contribution introduced to cancel the divergences coming from the Gibbons-Hawking boundary term (the second term above). The function $\hat{K}$ is defined implicitly by the solution of

$$R_{\alpha\beta} = \hat{K}_{\alpha\beta}\hat{K} - h^{\gamma\delta}\hat{K}_{\alpha\gamma}\hat{K}_{\delta\beta}, \qquad (3.2)$$

where $R_{\alpha\beta}$ is the Ricci tensor of the metric $h_{\alpha\beta}$ induced on ∂M. Thus this additional boundary term is determined locally by the induced metric on the boundary, in the spirit of the boundary counterterm approach to constructing actions for asymptotically AdS spaces [21]. Alternative actions for asymptotically flat spacetimes with a similar philosophy appeared previously in [22, 23]. See also [24] for related work.

To apply this prescription to asymptotically plane wave spacetimes, we first need to introduce a cutoff to make the different terms in the action finite. We will cut off the spacetime by introducing a boundary at some large radial distance, $r =$ constant. Our main focus will be on boundary terms associated with this boundary; as in the asymptotically

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flat case, there is a divergence associated with the Gibbons-Hawking boundary term on this surface due to the extrinsic curvature of the sphere, and we need to introduce an appropriate local boundary term to cancel it.

Although our focus is mainly on the r = constant boundary, to make the spacetime region we consider finite, we also need to introduce some cutoffs in the $x^{\pm}$ directions along the plane wave. The symmetry of the background under translations in x^- makes it natural to introduce cutoffs at two constant values of x^+, respecting this symmetry. In the simple case where μ_{ij} are constants, which includes the cases of most interest for holography, there is an additional symmetry under translations in x^+, which suggests it is natural to take the other cutoff to be at constant values of x^-, respecting this translation invariance. We will also discuss the calculation of the action for the general case where $\mu_{ij}(x^+)$ are not constants with this same cutoff. We will see that this choice of cutoff can give a satisfactory construction for an action even for general $\mu_{ij}(x^+)$, although there are some additional subtleties associated with the surfaces at constant x^-. However, one should bear in mind that there is no a priori justification for this choice of cutoff in the general case.

The action for the cutoff spacetime should contain a Gibbons-Hawking boundary term for each of these boundaries. In the case of the surfaces at x^+ = constant, there is a subtlety, as they are null surfaces, so the trace of the extrinsic curvature is not well-defined. However, this issue has been previously considered in [25], where it was shown that a suitable boundary term on a null boundary x^+ = constant is

$$\frac{1}{16\pi G}\int_{x^+=const} d^{d+1}\xi\sigma^\lambda\partial_\lambda x^+, \tag{3.3}$$

where $\sigma^\lambda = \frac{1}{\sqrt{-g}}\partial_\mu\left((-g)\, g^{\mu\lambda}\right)$, with g being the determinant of the metric on the full spacetime. We will adopt this prescription here. On the boundaries at x^- = constant, we consider just the usual Gibbons-Hawking boundary term.

On the boundary at r = constant, the Gibbons-Hawking boundary term gives a contribution which will diverge as we remove the cutoff. This divergence is associated with the intrinsic curvature of the boundary (the background plane wave spacetime has a flat spatial metric in the x^i directions, so the intrinsic and extrinsic curvatures of the r = constant boundary are related), so we can try to cancel this divergence by adding a Mann-Marolf counterterm contribution to the action on this boundary.

Thus, the action we consider is

$$\begin{aligned} S = &-\frac{1}{16\pi G}\int_M d^{d+2}x\sqrt{-g}R - \frac{1}{16\pi G}\int_{x^+=consts} d^{d+1}x\sigma^\lambda\partial_\lambda x^+ \\ &-\frac{1}{8\pi G}\int_{x^-=consts} d^{d+1}x\sqrt{h}K - \frac{1}{8\pi G}\int_{r=const} d^{d+1}x\sqrt{h}\left(K-\hat{K}\right), \end{aligned} \tag{3.4}$$

where by the integral over x^+ = constants we mean integrals over two surfaces at different values of x^+, with opposite orientations for the normal to the surface, and similarly for the integral over x^- = constants.

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Let us first of all consider the value of this action for the plane wave background (2.1). This is a vacuum solution, so $R = 0$. On the surface $x^+ = \text{constant}$,

$$\sigma^\lambda \partial_\lambda x^+ = \sigma^+ = \partial_\mu g^{\mu +} = 0, \tag{3.5}$$

as $g^{(0)++} = 0$ and $g^{(0)+-} = -1$. So the boundary term at $x^+ = \text{constant}$ vanishes. On the surface $x^- = \text{constant}$, if μ_{ij} are constant, the only non-zero component of $K_{\alpha\beta}$ is

$$K_{+i} = \frac{1}{2\sqrt{g^{(0)--}}} \partial_i g^{(0)}_{++}. \tag{3.6}$$

Since $h^{(0)+i} = 0$, this gives $K = 0$, and the boundary term at $x^- = \text{constant}$ vanishes as well.

In the more general case where $\mu_{ij}(x^+)$ depend on x^+, we have

$$K = K_{++} h^{(0)++} = \frac{1}{2\sqrt{g^{(0)--}}} \partial_+ g^{(0)}_{++} h^{(0)++}, \tag{3.7}$$

and at $x^- = \text{constant}$, $h^{(0)++} = 1/h^{(0)}_{++} = -1/(\mu_{ij}(x^+) x^i x^j)$. Hence, this $K \sim \mathcal{O}(r^{-1})$, and the contribution to the action is

$$S_- = -\frac{1}{8\pi G} \int_{x^- = const} K \sqrt{h} dx^+ d^d x^i \sim \mathcal{O}(r^d), \tag{3.8}$$

so this boundary will make a divergent contribution to the action as we remove the cutoff at large r. However, in the full action, there are two boundaries at constant x^- (at say $x^- = \pm x_0^-$), and they contribute with opposite signs because of the opposite orientations of the outward normals, so this term will cancel between the two boundaries, making no contribution to the total action.

Finally, the boundary at $r = \text{constant}$ is what we want to focus on, so let us be more explicit and set up the notation we will use later. Define coordinates on the boundary $x^\alpha = \{x^-, x^+, \theta^a\}$, so the boundary metric is

$$h_{\alpha\beta} = \begin{pmatrix} 0 & -1 & \vec{0} \\ -1 & -\mu_{ij} x^i x^j & \vec{0} \\ \vec{0} & \vec{0} & r^2 \hat{h}_{ab} \end{pmatrix}, \tag{3.9}$$

with determinant $h = -r^{2d-2}\hat{h}$, where $\hat{h}$ is the determinant of the unit metric on S^{d-1}. The normal vector to the boundary is $n_\nu = \delta^r_\nu$. The non-zero components of the extrinsic curvature are

$$K_{ab} = r\hat{h}_{ab}, \quad K_{++} = -\frac{\mu_{ij} x^i x^j}{r}, \tag{3.10}$$

so $K = \frac{d-1}{r}$. The Ricci tensor on the boundary is

$$R_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \vec{0} \\ 0 & R_{++} & \vec{0} \\ \vec{0} & \vec{0} & (d-2)\,\hat{h}_{ab} \end{pmatrix}. \tag{3.11}$$

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Solving (3.2) for $\hat{K}_{\alpha\beta}$, we find that the non-zero components are $\hat{K}_{ab} = r\hat{h}_{ab}$ and $\hat{K}_{++} = \frac{rR_{++}}{d-1}$, and so $\hat{K} = \frac{d-1}{r}$. Thus $K - \hat{K} = 0$ and hence there is no contribution to the action from the $r =$ constant surface.

Thus, we find that the on-shell action for the pure plane wave is zero. Note that the action vanishes for any plane wave solution, independent of the values of $\mu_{ij}(x^+)$.

3.1 Finiteness of the action

Next, we consider an arbitrary asymptotically plane wave solution satisfying our asymptotic falloff conditions, and show that the action of the solution will be finite. Since the metric g is still a solution of the vacuum equations, $R = 0$, so the bulk term still makes no contribution to the action. For the boundaries at constant x^+, as in the pure plane wave,

$$S_+ = -\frac{1}{16\pi G}\int_{x^+=const} dx^- \left(dx^i\right)^d \partial_\mu g^{(1)\mu+}. \tag{3.12}$$

In the gauge we have chosen, $g^{++} = 0$, $g^{+-} = 1$, and $g^{+i} = 0$, so this term still vanishes.

For the boundaries at constant x^-, the contributions to the extrinsic curvature at linear order in the departure of the metric from the plane wave are

$$K = K^{(0)}_{++} h^{(1)++} + K^{(0)}_{+i} h^{(1)+i} + K^{(1)}_{++} h^{(0)++} + K^{(1)}_{ij} h^{(0)ij}. \tag{3.13}$$

On these boundaries, we have $h^{(1)++} \sim \mathcal{O}(r^{-d})$, $h^{(1)+i} \sim \mathcal{O}(r^{1-d})$, and

$$K^{(1)}_{++} = -\frac{1}{2}\frac{g^{(0)+-}}{\sqrt{g^{(0)--}}}\partial_+ g^{(1)}_{++} - \frac{1}{2}\frac{g^{(0)+-}g^{(1)--}}{(g^{(0)--})^{3/2}}\partial_+ g^{(0)}_{++} + \frac{1}{2}\sqrt{g^{(0)--}}\partial_- g^{(1)}_{++}, \tag{3.14}$$

$$K^{(1)}_{ij} = -\frac{1}{2}\frac{g^{(0)+-}}{\sqrt{g^{(0)--}}}\left(\partial_j g^{(1)}_{i+} + \partial_i g^{(1)}_{j+} - \partial_+ g^{(1)}_{ij}\right) + \frac{1}{2}\sqrt{g^{(0)--}}\partial_- g^{(1)}_{ij}. \tag{3.15}$$

Thus, the terms which are independent of x^- will give a contribution to $K \sim \mathcal{O}(r^{1-d})$. This will make a divergent contribution to the integral over a single boundary, $S_- \sim \mathcal{O}(r^2)$. However, as in the action for the pure plane wave, this divergence cancels between the two boundaries, so for asymptotically plane wave solutions which are independent of x^-, the contribution to the action from these boundaries vanishes.

We require that any terms depending on x^- fall off at large x^-. This implies in particular that there cannot be any linear dependence on x^- near these boundaries, so the part of the components $g^{(1)}_{\mu\nu}$ involving x^- will fall off faster than the part that is independent of x^- by a factor of $1/r^4$. The contribution of the x^--dependent part of $g^{(1)}_{\mu\nu}$ to the terms in K that do not involve explicit derivatives ∂_- will then be $O(r^{-d-3})$. Thus the contribution to the action from this part of K is finite, and will go to zero as we take the cutoff in x^- to infinity. There are terms in $K^{(1)}_{++}$ and $K^{(1)}_{ij}$ which involve explicit derivatives ∂_-: these make a contribution $K \sim \mathcal{O}(r^{-d-1})$, giving a contribution to the integral S_- which is logarithmically divergent at large r. However, this contribution comes with some negative power of x^-, so if we take the boundaries at constant x^- to infinity at the same time as we take the boundary at large r to infinity, this contribution will go to zero. This dependence

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on the order of limits is not entirely satisfactory, but it allows us to define a finite action. It does not seem to conceal any particularly interesting deeper issues.

Finally, we consider the boundary at $r=$ constant. We can write the linear order contribution to the boundary term in our gauge as

$$K^{(1)}-\hat{K}^{(1)}=K^{(1)}_{\alpha\beta}h^{(0)\alpha\beta}-\hat{K}^{(1)}_{\alpha\beta}h^{(0)\alpha\beta}. \tag{3.16}$$

As $\sqrt{h}\sim\mathcal{O}(r^{d-1})$, we need $K^{(1)}-\hat{K}^{(1)}\sim\mathcal{O}(r^{1-d})$ to have a finite action. For the term involving the extrinsic curvature,

$$K^{(1)}_{\alpha\beta}=g^{(1)rr}K^{(0)}_{\alpha\beta}+\frac{1}{2}\left(g^{(1)}_{\beta r,\alpha}+g^{(1)}_{r\alpha,\beta}-g^{(1)}_{\alpha\beta,r}\right), \tag{3.17}$$

and substituting for $g^{(1)}_{\alpha\beta}$ it is easy to show that this term is $\mathcal{O}(r^{1-d})$.

To evaluate $\hat{K}^{(1)}_{\alpha\beta}$, we linearize (3.2) to give

$$R^{(1)}_{\alpha\beta}=\hat{K}^{(1)}_{\gamma\delta}L^{(0)\gamma\delta}_{\alpha\beta}+\left(\hat{K}^{(0)}_{\alpha\beta}\hat{K}^{(0)}_{\gamma\delta}-\hat{K}^{(0)}_{\alpha\gamma}\hat{K}^{(0)}_{\beta\delta}\right)h^{(1)\gamma\delta}, \tag{3.18}$$

where[4]

$$L^{(0)\gamma\delta}_{\alpha\beta}=h^{\gamma\delta}\hat{K}_{\alpha\beta}+\frac{1}{2}\left(\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}\hat{K}+\delta^{\gamma}_{\beta}\delta^{\delta}_{\alpha}\hat{K}\right)-\frac{1}{2}\left(\delta^{\gamma}_{\alpha}\hat{K}^{\delta}_{\ \beta}+\delta^{\gamma}_{\beta}\hat{K}^{\delta}_{\ \alpha}+\delta^{\delta}_{\alpha}\hat{K}^{\gamma}_{\ \beta}+\delta^{\delta}_{\beta}\hat{K}^{\gamma}_{\ \alpha}\right). \tag{3.19}$$

Inverting this will give us an expression for $\hat{K}^{(1)}_{\alpha\beta}$,

$$h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta}=M^{(0)\gamma\delta}\left(R^{(1)}_{\gamma\delta}-\left(\hat{K}^{(0)}_{\alpha\beta}\hat{K}^{(0)}_{\gamma\delta}-\hat{K}^{(0)}_{\alpha\gamma}\hat{K}^{(0)}_{\beta\delta}\right)h^{(1)\alpha\beta}\right), \tag{3.20}$$

where $M^{\gamma\delta}=h^{\alpha\beta}\left(L^{-1}\right)_{\alpha\beta}^{\ \ \gamma\delta}$. Recall that the non-zero components in $\hat{K}^{(0)}_{\alpha\beta}$ are $\hat{K}^{(0)}_{++}$ and $\hat{K}^{(0)}_{ab}$, and note that in our gauge $h^{(1)++}=0$ on the $r=$ constant boundary. We thus have

$$h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta}=M^{(0)\alpha\beta}R^{(1)}_{\alpha\beta}-M^{(0)ab}(\hat{K}^{(0)}_{ab}\hat{K}^{(0)}_{cd}-\hat{K}^{(0)}_{ac}\hat{K}^{(0)}_{bd})h^{(1)cd}. \tag{3.21}$$

A lengthy explicit calculation gives that the only non-zero components of $M^{(0)\gamma\delta}$ are

$$M^{(0)+-}\sim\mathcal{O}(r),\quad M^{(0)--}\sim\mathcal{O}(r^2),\quad M^{(0)ab}=\frac{1}{2(d-2)r}\hat{h}^{ab}=\frac{r}{2(d-2)}h^{ab}. \tag{3.22}$$

For the second term in (3.21), we have $\hat{K}^{(0)}_{ab}\sim\mathcal{O}(r)$, and $h^{(1)cd}\sim\mathcal{O}(r^{-d})$, so this term is $\mathcal{O}(r^{1-d})$. For the first term, we express $R^{(1)}_{\alpha\beta}$ by the analogue of (2.3),

$$R^{(1)}_{\alpha\beta}=-\frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\alpha}D^{(0)}_{\beta}h^{(1)}_{\gamma\delta}-\frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{\delta}h^{(1)}_{\alpha\beta}+h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{(\alpha}h^{(1)}_{\beta)\delta}, \tag{3.23}$$

where D_α is the covariant derivative compatible with $h_{\alpha\beta}$. Using this expression we can see that $R^{(1)}_{+-}\sim\mathcal{O}(r^{-d})$, $R^{(1)}_{--}\sim\mathcal{O}(r^{-d-2})$, and $R^{(1)}_{ab}\sim\mathcal{O}(r^{2-d})$, so the first term also makes a finite contribution (in addition, many of these terms will actually be total derivatives, which make no contribution to the action).

Thus, we conclude that the on-shell action is finite for the asymptotically plane wave spacetimes.

[4]Note that we define $L^{(0)\gamma\delta}_{\alpha\beta}$ so that it is symmetric in both pairs of indices, so this is slightly different from the corresponding expression in [18].

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3.2 Variations of the action

In addition to being finite on-shell, we would like to see that $\delta S = 0$ for arbitrary variations about a solution of the equations of motion. The variation of the usual Einstein-Hilbert plus Gibbons-Hawking action would give a boundary term

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int \sqrt{-h}\pi^{\alpha\beta}\delta h_{\alpha\beta}, \tag{3.24}$$

where $\pi^{\alpha\beta} = K^{\alpha\beta} - h^{\alpha\beta}K$. On the boundaries at $x^+ =$ constant and $x^- =$ constant, we have just this term. Therefore if we require $\delta h_{\alpha\beta} = 0$ on these boundaries, they will make no contribution to the variation of the action. This is a reasonable boundary condition if we think of these as fixed cutoffs; that is, if we will keep the coordinate position of the cutoff fixed as we vary the metric, and do not intend to eventually send the cutoff to infinity. This is certainly an appropriate approach for the $x^+ =$ constant boundary. In some cases, however, it is more appropriate to eventually remove the cutoff on x^-. For this purpose, we could imagine relaxing this condition to require only that $\delta h_{\alpha\beta}$ decays as we go to large x^-. Since the background metric is independent of x^-, any $\delta h_{\alpha\beta}$ which goes to zero at large x^- will produce a contribution to δS which vanishes as we remove the cutoff on x^-. Thus, there is no problem with the variation of the action involving these boundaries.

We turn to the contribution to the variation of the action from the boundary at $r =$ constant, where we only want to require that the variation $\delta h_{\alpha\beta}$ falls off as quickly as $g^{(1)}_{\alpha\beta}$. On the $r =$ constant boundary, we have the above boundary contribution from the Einstein-Hilbert plus Gibbons-Hawking action, and we have the contribution coming from the variation of the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G}\int \sqrt{-h}\left(-\frac{1}{2}\hat{K}h^{\alpha\beta}\delta h_{\alpha\beta} + \hat{K}_{\alpha\beta}\delta h^{\alpha\beta} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right). \tag{3.25}$$

To determine $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$, we need to use the analogue of (3.18) for variations to write

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{\gamma\delta}\left(\delta R_{\gamma\delta} - \left(\hat{K}_{\alpha\beta}\hat{K}_{\gamma\delta} - \hat{K}_{\alpha\gamma}\hat{K}_{\beta\delta}\right)\delta h^{\alpha\beta}\right), \tag{3.26}$$

where $\delta R_{\gamma\delta}$ is given in terms of $\delta h_{\alpha\beta}$ by

$$\delta R_{\alpha\beta} = -\frac{1}{2}h^{\gamma\delta}D_\alpha D_\beta\delta h_{\gamma\delta} - \frac{1}{2}h^{\gamma\delta}D_\gamma D_\delta\delta h_{\alpha\beta} + h^{\gamma\delta}D_\gamma D_{(\alpha}\delta h_{\beta)\delta}. \tag{3.27}$$

The variation can be taken to respect our choice of gauge, so $\delta h_{-\alpha} = 0$. Thus, we only need to consider the variations δh_{++}, δh_{+a} and δh_{ab}.

Let's consider first just δh_{++} non-zero. The term in δS_{EH+GH} involving δh_{++} is trivially zero, as $\pi^{++} = 0$ with our choice of gauge. For the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G}\int \sqrt{-h}\left(\hat{K}^{++}\delta h_{++} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right). \tag{3.28}$$

This expression involves the full metric of the asymptotically plane wave solution we are considering. For each term, we will explicitly calculate the result for the leading non-zero

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contribution (coming either from $g^{(0)}$ or $g^{(1)}$). Higher-order terms are suppressed, so if the first term gives zero contribution to the variation of the action, we do not need to consider higher orders. In the first term in (3.28), solving for $\hat{K}^{(1)++}$ using (3.18) gives $\hat{K}^{(1)++} \sim \mathcal{O}(r^{-d-1})$, and $\delta h_{++} \sim \mathcal{O}(r^{4-d})$, so $\hat{K}^{++}\delta h_{++} \sim \mathcal{O}(r^{3-2d})$, and the first term in the integral is $\mathcal{O}(r^{2-d})$, which vanishes for $d \geq 3$. For the second term, we use (3.26), where there will be a zeroth-order contribution to the first term and a first-order contribution to the second term. From (3.27), we find that δh_{++} gives only δR_{++}, δR_{+-} and δR_{+a} non-zero. Using our previous calculation of the components $M^{(0)\alpha\beta}$, we then have

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{(0)+-}\delta R^{(0)}_{+-} - M^{(0)ab}\hat{K}^{(0)}_{ab}\hat{K}^{(1)++}\delta h_{++}. \tag{3.29}$$

Now $\delta R^{(0)}_{+-} = -\frac{1}{2}h^{(0)+-}\partial_-\partial_-\delta h_{++} \sim \mathcal{O}(r^{-d})$, so the first term is $\mathcal{O}(r^{1-d})$. Together with the factor of $\sqrt{-h}$ in the integral, this would give a finite contribution to the variation. However, this leading-order term is a total derivative, because $h^{(0)}_{\alpha\beta}$ is independent of x^-, so it makes no contribution. Higher-order contributions from this term would not be a total derivative, but they are suppressed by further powers of r, so their contribution to the action vanishes in the large r limit. The second term is of the same form as the contribution considered above, giving a contribution $h^{\alpha\beta}\hat{K}_{\alpha\beta} \sim \mathcal{O}(r^{3-2d})$. Thus all the terms coming from δh_{++} vanish in the large r limit.

We now evaluate terms involving δh_{a+}. We find

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int \sqrt{-h}\pi^{a+}\delta h_{a+}. \tag{3.30}$$

At linear order, $\pi^{a+} \sim h^{ab}\partial_- h_{br} \sim \mathcal{O}(r^{-d-1})$, and $\delta h_{a+} \sim \mathcal{O}(r^{4-d})$, so this term is vanishing for $d \geq 3$. For the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G}\int \sqrt{-h}\left(\hat{K}^{a+}\delta h_{a+} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right), \tag{3.31}$$

and (3.18) gives $\hat{K}^{(1)a+} \sim \mathcal{O}(r^{-d-1})$, so the first term also vanishes for $d \geq 3$. In the second term, having just δh_{a+} gives us all components of $\delta R_{\alpha\beta}$ except δR_{--} non-zero. Using (3.26) and our previous calculation of the components $M^{(0)\alpha\beta}$, we then have

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{(0)+-}\delta R^{(0)}_{+-} + M^{(0)ab}\delta R^{(0)}_{ab} - M^{(0)ab}\hat{K}^{(0)}_{ab}\hat{K}^{(1)a+}\delta h_{a+}. \tag{3.32}$$

We have $\delta R^{(0)}_{+-} = \frac{1}{2}h^{(0)cb}D^{(0)}_b\partial_-\delta h_{+c} \sim \mathcal{O}(r^{-d})$, and $\delta R^{(0)}_{ab} = \frac{1}{2}h^{(0)+-}\partial_- D^{(0)}_b\delta h_{a+} \sim \mathcal{O}(r^{2-d})$. Thus, both of the first two terms in $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$ would make finite contributions to the variation of the action. However, as they involve ∂_-, they are total derivatives, so they actually make zero contribution. As in the previous case when we analysed terms involving δh_{++}, higher-order contributions from this term would not be a total derivative, but they are suppressed by further powers of r, so their contribution to the action vanishes in the large r limit. The final term in $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$ is of the same form as the contribution to the variation coming from $\hat{K}^{a+}\delta h_{a+}$, so it goes like $\mathcal{O}(r^{3-2d})$, and all the terms in the variation of the action coming from δh_{a+} vanish in the large r limit.

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Finally we consider terms involving δh_{ab}. We find

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int \sqrt{-h}\pi^{ab}\delta h_{ab}, \tag{3.33}$$

and since $\pi^{ab} \sim \mathcal{O}(r^{-3})$ and $\delta h_{ab} \sim \mathcal{O}(r^{4-d})$, this gives an r^0 term which does not vanish in the large r limit. This term needs to be cancelled by a corresponding term coming from δS_{MM}. The latter is

$$\begin{aligned}\delta S_{MM} &= \frac{1}{8\pi G}\int \sqrt{-h}\left(-\frac{1}{2}\hat{K}h^{\alpha\beta}\delta h_{\alpha\beta} + \hat{K}_{\alpha\beta}\delta h^{\alpha\beta} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right)\\ &= \frac{1}{8\pi G}\int \sqrt{-h}\left(\frac{1}{2}\hat{\pi}^{ab}\delta h_{ab} + \frac{1}{2}\hat{K}^{ab}\delta h_{ab} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right),\end{aligned} \tag{3.34}$$

where $\hat{\pi}^{ab} = \hat{K}^{ab} - h^{ab}\hat{K}$. To zeroth order, $\hat{\pi}^{(0)ab} = \pi^{(0)ab}$, so the first term in (3.34) cancels the non-zero contribution from (3.33). However, the second term in (3.34) also has a non-zero leading order part, so we need to see that this can be cancelled by a contribution from the final term. Considering the variation δh_{ab},

$$\begin{aligned}h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = &\, M^{(0)+-}\delta R^{(0)}_{+-} + M^{(0)--}\delta R^{(0)}_{--} + M^{(0)ab}\delta R^{(0)}_{ab}\\ &-M^{(0)ab}\left(\hat{K}^{(0)}_{ab}\hat{K}^{(0)}_{cd} - \hat{K}^{(0)}_{ac}\hat{K}^{(0)}_{bd}\right)\delta h^{cd}.\end{aligned} \tag{3.35}$$

The terms involving $\delta R_{\alpha\beta}$ give finite contributions which are total derivatives, as before. For the first two terms,

$$\delta R^{(0)}_{+-} = h^{(0)ab}D^{(0)}_{+}\partial_{-}\delta h_{ab} \sim \mathcal{O}(r^{-d}), \quad \delta R^{(0)}_{--} = h^{(0)ab}\partial_{-}\partial_{-}\delta h_{ab} \sim \mathcal{O}(r^{2-d}), \tag{3.36}$$

and these are total derivatives because $h^{(0)}_{\alpha\beta}$ is independent of x^{-}. For the other term, $\delta R^{(0)}_{ab} \sim \mathcal{O}(r^{2-d})$ involves covariant derivatives with respect to the unit metric on S^{d-1}, $\hat{h}_{ab}$, and this term is a total derivative because the only θ_a dependence in the terms multiplying $\delta R^{(0)}_{ab}$ is through the covariantly constant metric $\hat{h}_{ab}$. As in the previous two cases, higher-order contributions from these terms would not be total derivatives, but they are suppressed by further powers of r, so their contribution to the action vanishes in the large r limit. We are then left with evaluating the last term in (3.35). Using $\hat{K}^{(0)}_{ab} = r\hat{h}_{ab}$ and $M^{(0)ab} = \frac{1}{2(d-2)}\hat{h}^{ab}$,

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} \to -M^{(0)ab}\left(\hat{K}^{(0)}_{ab}\hat{K}^{(0)}_{cd} - \hat{K}^{(0)}_{ac}\hat{K}^{(0)}_{bd}\right)\delta h^{cd} = -\frac{1}{2}r\hat{h}^{ab}\delta h_{ab} = -\frac{1}{2}\hat{K}^{(0)ab}\delta h_{ab}. \tag{3.37}$$

This will cancel with the leading order part of the second term in (3.34), leaving us with no finite contributions to the variation of the action in the large r limit.

Thus, this action gives a well-defined variational principle for our class of asymptotically plane wave spacetimes.

4. Conclusions

In this paper, we have given a definition of asymptotically plane wave spacetimes which is consistent with the known exact solutions, and constructed a well-behaved action principle

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for asymptotically plane wave solutions of the vacuum Einstein equations, following the work of [14]. The definition of asymptotically plane wave solutions is valid for any solution which asymptotically approaches a vacuum plane wave. For the action, we considered only the pure vacuum action; it would be interesting to extend this work to include appropriate matter fields. It is also interesting to ask if there are non-trivial physically relevant examples to which our ideas apply.[5] For the asymptotically plane wave boundary conditions, (2.13) provides such an example, but this is not a pure vacuum solution, so our discussion of the action does not apply to it. A more trivial example is provided by some pp-wave solutions. For example, consider the vacuum pp-wave metric

$$ds^2 = -2dx^+dx^- - A(x^+, x^i)\left(dx^+\right)^2 + \delta_{ij}dx^i dx^j \tag{4.1}$$

with $\partial_i\partial^i A = 0$. If $A(x^+, x^i) \to \mu_{ij}(x^+)x^i x^j + \mathcal{O}(r^{4-d})$ as $r \to \infty$, this solution is asymptotically plane wave according to our definition, and the action we have defined will be finite for it. However, this is a rather trivial example, and it would be interesting to construct solutions really corresponding to localised sources in an asymptotically plane wave background, and we hope to return to this question in future work.

We have just demonstrated that the action is well-behaved; an obvious extension of this work would be to go on to construct a boundary stress tensor $\langle T_{\alpha\beta}(x^+, x^-, \theta^a)\rangle$, as was done for the asymptotically flat case in [14] and for the linear dilaton case in [18]. This could then be used to calculate conserved quantities. The fact that different components of $g^{(1)}$ fall off at different rates at large r may lead to some interesting subtleties in extending the previous work to this case; perhaps, as in the asymptotically flat case, there will be more than one stress tensor, associated with different orders in the asymptotic expansion.

A central motivation for work in this direction is to better understand holography for the plane wave. In [17], it was argued that a holographic dual of asymptotically flat space could be constructed on the hyperbola at spatial infinity, calculating two-point functions in the holographic dual from variations of the action. It is possible that similar ideas could be applied in this case, but there is no obvious connection between this notion of holography and the known example. String theory on the plane wave obtained from the Penrose limit of $\mathrm{AdS}_5 \times S^5$ is dual to a quantum mechanics, so it has observables depending on a single coordinate, whereas if we were to construct a boundary stress tensor $\langle T_{\alpha\beta}(x^+, x^-, \theta^a)\rangle$ or two-point functions on the boundary at large r from our action, we would expect them to generically depend on all the boundary coordinates. Our remarks in section 2.2 on the relation between our notion of asymptotically plane wave and the conformal boundary of the maximally supersymmetric plane wave suggest that the boundary at large r we have focused on is not, at least, the whole story. To understand the relation to holography, we probably need to study the boundaries at constant x^- in more detail, and the information coming just from large r may be misleading.

This asymptotically plane wave example thus seems to have some interesting differences compared to previous attempts to study holography for more general spacetimes, and we

[5]We thank the referee for raising this point.

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hope this work will shed some useful light on the relation between the bulk action and the holographic dual theory for other spacetimes, which in general remains to be worked out.

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